Solution of linearized Fokker - Planck equation for incompressible fluid

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Abstract

In this work we construct algebraic equation for elements of spectrum of linearized Fokker - Planck differential operator for incompressible fluid. We calculate roots of this equation using simple numeric method. For all these roots real part is positive, that is corresponding solutions are damping. Eigenfunctions of linearized Fokker - Planck differential operator for incompressible fluid are expressed as linear combinations of eigenfunctions of usual Fokker - Planck differential operator. Poisson's equation for pressure is derived from incompressibility condition. It is stated, that the pressure could be totally eliminated from dynamics equations. The Cauchy problem setup and solution method is presented. The role of zero pressure solutions as eigenfunctions for confluent eigenvalues is emphasized.

Keywords

Fokker-Planck equation, incompressible fluid, linear operator spectrum

1 Introduction

In our previous work [1] we derived linearized Fokker - Planck equation for incompressible fluid

$$\int_V n dv_x dv_y dv_z = 0. \tag{1}$$

$$\frac{\partial n}{\partial t} + v_j \frac{\partial n}{\partial x_j} - \alpha \ \frac{\partial}{\partial v_j}(v_j n) + \left(\frac{\alpha}{k}\right) \ \left(\frac{\alpha}{2\pi k}\right)^{3/2} \exp\left[-\frac{\alpha}{2k} v_j v_j\right] \ v_k \frac{\partial p}{\partial x_k} = k \ \frac{\partial^2 n}{\partial v_j \partial v_j}. \tag{2}$$

where

$n = n(t, x_1, x_2, x_3, v_1, v_2, v_3)$ - density;

$p = p(t, x_1, x_2, x_3)$ - pressure;

t - time variable;

x_1, x_2, x_3 - space coordinates;

v_1, v_2, v_3 - velocities;

α - coefficient of damping;

k - coefficient of diffusion.

No attempt was made to solve this equation.

Peculiarity of this equation consists in the fact, that for two unknown variables n and p we have only one differential equation. This is enough, because there is additional normalization requirement (1) on n variable and p variable depends only on space coordinates x, y, z and time t.

In [2] some simple solutions of nonlinear equation were studied. Particularly flow with zero pressure is of interest for our present studies, because for this flow the source of nonlinearity is absent. This flow solves both nonlinear and linear equations.

Present work is devoted to solution of linearized equation. We consider only the case of parallelepiped with opposite sides identified (i.e. "periodic boundary conditions"). We try to formulate and solve Cauchy problem for linearized equation.

2 Fourier decomposition of solution

We know the form of Cauchy problem solution for the case of usual Fokker-Planck equation (see [3]).

$$n(t, x_j, v_j) = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} A_{m_1 m_2 m_3 p_1 p_2 p_3}(t)\ \phi_{m_1 m_2 m_3 p_1 p_2 p_3}, \tag{3}$$

where eigenfunctions of usual Fokker - Planck operator are

$$\phi_{m_1 m_2 m_3 n_1 n_2 n_3} = \prod_{j=1}^{j=3} \exp\left(2\pi i \frac{m_j}{a_j}(x_j - \frac{v_j}{\alpha})\right) \exp\left(-\frac{\alpha}{2k} v_j^2\right) H_{n_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j}\right)\right). \tag{4}$$

It is only natural to seek solution of the present problem in the same form. We need only add expression for the new p variable

$$p(t, x_j) = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \sum_{m_3=-\infty}^{+\infty} P_{m_1 m_2 m_3}(t) \prod_{j=1}^{j=3} \exp\left(2\pi i \frac{m_j}{a_j} x_j\right), \tag{5}$$

We try to represent the coefficient before $\frac{\partial p}{\partial x_k}$ in equation (2) in the same way as a sum of Fourier series

$$\left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2} \prod_{j=1}^{j=3} \exp\left(2\pi i \frac{m_j}{a_j} x_j\right) \exp\left[-\frac{\alpha}{2k} v_j v_j\right]\ v_k = \tag{6}$$

$$= \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{p_3=0}^{\infty} B_{m_1 m_2 m_3 n_1 n_2 n_3}(k) \phi_{m_1 m_2 m_3 n_1 n_2 n_3}.$$

In expressions (3, 5, 6) we introduced following Fourier coefficients:

$A_{m_1m_2m_3p_1p_2p_3}(t)$ - coefficients of decomposition unknown variable n;

$P_{m_1m_2m_3}(t)$ - coefficients of decomposition of unknown variable p;

$B_{m_1m_2m_3n_1n_2n_3}(k)$ - coefficients of decomposition of known variable, which represent coefficient before p gradient. Values of B are presented below.

Using these coefficients, we rewrite equation (2) as

$$\frac{d}{dt}A_{m_1m_2m_3n_1n_2n_3}(t) + \sum_{j=1}^{j=3}\left[\alpha n_j + k\ \left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\ A_{m_1m_2m_3n_1n_2n_3}(t) = \tag{7}$$

$$= 2\pi i\frac{m_k}{a_k}B_{m_1m_2m_3n_1n_2n_3}(k)P_{m_1m_2m_3}(t)\ .$$

We reduced the partial differential equation (2) to system of ordinary differential equations for Fourier coefficients. To proceed with solution, we need expressions for known coefficients $B_{m_1m_2m_3n_1n_2n_3}(k)$.

3 Expressions for Fourier coefficients

In this auxiliary section we find explicit expressions for Fourier coefficients of some known functions of velocities. These functions are products of multipliers, which depend only on one independent variable. Therefore we can consider only one velocity variable in this section.

We start from definitions

$$\phi_{mn} = \exp\left(-\frac{2\pi i m}{\alpha a}v\right)\ \exp\left(-\frac{\alpha}{2k}v^2\right)H_n\left(\sqrt{\frac{\alpha}{2k}}\left(v+\frac{4\pi i m k}{\alpha^2 a}\right)\right). \tag{8}$$

$$\psi_{mn} = \exp\left(-\frac{2\pi i m}{\alpha a}v\right)\ H_n\left(\sqrt{\frac{\alpha}{2k}}\left(v+\frac{4\pi i m k}{\alpha^2 a}\right)\right). \tag{9}$$

Functions ϕ_{mn} and ψ_{mn} are of course orthogonal (see [3])

$$\int_{-\infty}^{\infty}\phi_{mp}\ \ \psi_{mq}dv = \exp\left[-\frac{\alpha}{2k}\ \left(\frac{4\pi m k}{\alpha^2 a}\right)^2\right]\ \sqrt{\frac{2\pi k}{\alpha}}\delta_{pq}(-2)^p p!. \tag{10}$$

Let us find Fourier coefficients for following functions:

$$\exp\left[-\frac{\alpha}{2k}v^2\right] = \sum_{n=0}^{\infty}a_{mn}\phi_{mn}. \tag{11}$$

$$\exp\left[-\frac{\alpha}{2k}v^2\right]\ \ v = \sum_{n=0}^{\infty}b_{mn}\phi_{mn}. \tag{12}$$

To find these coefficients, we need to calculate integrals

$$a_{mn} = \exp\left[\frac{\alpha}{2k}\ \left(\frac{4\pi m k}{\alpha^2 a}\right)^2\right]\ \sqrt{\frac{\alpha}{2\pi k}}\frac{1}{(-2)^n n!}\int_{-\infty}^{\infty}\exp\left[-\frac{\alpha}{2k}v^2\right]\ \ \psi_{mn}dv. \tag{13}$$

$$b_{mn}=\exp\left[\frac{\alpha}{2k}\ \left(\frac{4\pi mk}{\alpha^2 a}\right)^2\right]\ \sqrt{\frac{\alpha}{2\pi k}}\frac{1}{(-2)^n n!}\int_{-\infty}^{\infty}\exp\left[-\frac{\alpha}{2k}v^2\right]\ v\ \psi_{mn}dv. \tag{14}$$

We calculated these integrals in our previous work [2]

$$\int_{-\infty}^{\infty}\exp\left[-\frac{\alpha}{2k}v^2\right]\ \psi_{mn}dv= \tag{15}$$

$$=\int_{-\infty}^{\infty}\exp\left[-\frac{\alpha}{2k}v^2\right]\ \exp\left(-\frac{2\pi im}{\alpha a}v\right)\ H_n\left(\sqrt{\frac{\alpha}{2k}}\left(v+\frac{4\pi imk}{\alpha^2 a}\right)\right)\ dv=$$

$$=\sqrt{\frac{2\pi k}{a}}\left(\frac{2k}{\alpha}\right)^{n/2}\exp\left[-\frac{k}{2\alpha}\left(\frac{2\pi m}{\alpha a}\right)^2\right]\ \left(\frac{2\pi im}{\alpha a}\right)^n.$$

$$\int_{-\infty}^{\infty}\exp\left[-\frac{\alpha}{2k}v^2\right]\ v\ \psi_{mn}dv= \tag{16}$$

$$=\int_{-\infty}^{\infty}\exp\left[-\frac{\alpha}{2k}v^2\right]\ v\ \exp\left(-\frac{2\pi im}{\alpha a}v\right)\ H_n\left(\sqrt{\frac{\alpha}{2k}}\left(v+\frac{4\pi imk}{\alpha^2 a}\right)\right)\ dv=$$

$$=\sqrt{\frac{2\pi k}{a}}\left(\frac{2k}{\alpha}\right)^{n/2}\exp\left[-\frac{k}{2\alpha}\left(\frac{2\pi m}{\alpha a}\right)^2\right]\ \left[-\frac{k}{\alpha}\left(\frac{2\pi im}{\alpha a}\right)^{n+1}+n\ \left(\frac{2\pi im}{\alpha a}\right)^{n-1}\right].$$

Thus we get following expressions for Fourier coefficients

$$\boxed{a_{mn}=\exp\left[6\frac{k}{\alpha}\left(\frac{\pi m}{\alpha a}\right)^2\right]\ \frac{1}{2^n n!}\left(\frac{2k}{\alpha}\right)^{n/2}\left(\frac{2\pi im}{\alpha a}\right)^n.} \tag{17}$$

$$\boxed{b_{mn}=\exp\left[6\frac{k}{\alpha}\left(\frac{\pi m}{\alpha a}\right)^2\right]\ \frac{1}{2^n n!}\left(\frac{2k}{\alpha}\right)^{n/2}\left[-\frac{k}{\alpha}\left(\frac{2\pi im}{\alpha a}\right)^{n+1}+n\ \left(\frac{2\pi im}{\alpha a}\right)^{n-1}\right].} \tag{18}$$

4 Dynamics of Fourier coefficients

In this section we substitute expressions for known coefficients in terms of $a_{m_2n_2}$ and $b_{m_1n_1}$ coefficients, which we find in the last section, to the main equation (7). Namely, we use expressions

$$B_{m_1m_2m_3n_1n_2n_3}(1)=\left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\ b_{m_1n_1}a_{m_2n_2}a_{m_3n_3}. \tag{19}$$

$$B_{m_1m_2m_3n_1n_2n_3}(2)=\left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\ a_{m_1n_1}b_{m_2n_2}a_{m_3n_3}. \tag{20}$$

$$B_{m_1m_2m_3n_1n_2n_3}(3)=\left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\ a_{m_1n_1}a_{m_2n_2}b_{m_3n_3}. \tag{21}$$

Then (7) reads

$$\frac{d}{dt}A_{m_1m_2m_3n_1n_2n_3}(t)+\sum_{j=1}^{j=3}\left[\alpha n_j+k\ \left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\ A_{m_1m_2m_3n_1n_2n_3}(t)= \tag{22}$$

$$=2\pi i\ \left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\left(\frac{m_1}{a_1}b_{m_1n_1}a_{m_2n_2}a_{m_3n_3}+\frac{m_2}{a_2}a_{m_1n_1}b_{m_2n_2}a_{m_3n_3}+\right.$$

$$\left.+\frac{m_3}{a_3}a_{m_1n_1}a_{m_2n_2}b_{m_3n_3}\right)\ P_{m_1m_2m_3}(t)\ .$$

or

$$\frac{d}{dt}A_{m_1m_2m_3n_1n_2n_3}(t)+\sum_{j=1}^{j=3}\left[\alpha n_j+k\ \left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\ A_{m_1m_2m_3n_1n_2n_3}(t)= \tag{23}$$

$$=2\pi i\ \left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}a_{m_1n_1}a_{m_2n_2}a_{m_3n_3}\ \left(\frac{m_1}{a_1}b_{\frac{m_1n_1}{a_{m_1n_1}}}+\frac{m_2}{a_2}b_{\frac{m_2n_2}{a_{m_2n_2}}}+\right.$$

$$\left.+\frac{m_3}{a_3}b_{\frac{m_3n_3}{a_{m_3n_3}}}\right)\ P_{m_1m_2m_3}(t)\ .$$

This is main equation, which describe dynamics of Fourier coefficients. We delay actual substitution of $a_{m_in_j}$ and $b_{m_in_j}$ until (31).

5 Incompressibility condition

We use the incompressibility condition (1) to eliminate $P_{m_1m_2m_3}(t)$ from (23).

(1) and (3) imply

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}A_{m_1m_2m_3n_1n_2n_3}(t)\ \int_V\phi_{m_1m_2m_3n_1n_2n_3}dv_xdv_ydv_z=0. \tag{24}$$

Let us denote

$$c_{mn}=\int_{-\infty}^{\infty}\exp\left[-\frac{\alpha}{2k}v^2\right]\ \psi_{mn}dv=\int_{-\infty}^{\infty}\phi_{mn}dv= \tag{25}$$

$$=\sqrt{\frac{2\pi k}{a}}\left(\frac{2k}{\alpha}\right)^{n/2}\exp\left[-\frac{k}{2\alpha}\left(\frac{2\pi m}{\alpha a}\right)^2\right]\ \left(\frac{2\pi i m}{\alpha a}\right)^n.$$

So incompressibility condition is equivalent to following equation

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}A_{m_1m_2m_3n_1n_2n_3}(t)\ =0. \tag{26}$$

Let us suppose, that coefficients $A_{m_1m_2m_3n_1n_2n_3}(t)$ satisfy (26) at the moment t. They must satisfy this equation at the next moment $t+dt$. Values of A at the next moment are defined by dynamics equation (23), which contains besides $A(t)$ also pressure P. Therefore incompressibility condition, written for the next moment, will give us equation for pressure P. Derivation of this equation is rather long procedure, which ends in equation (42).

Differentiate (26) on time t and get

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}A'_{m_1m_2m_3n_1n_2n_3}(t) \ = 0. \tag{27}$$

Let us find $A'_{m_1m_2m_3n_1n_2n_3}(t)$ from (22) and substitute this value to (27)

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}\sum_{j=1}^{j=3}\left[\alpha n_j + k\ \left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\ A_{m_1m_2m_3n_1n_2n_3}(t) = \tag{28}$$

$$= 2\pi i\ \left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}\times$$

$$\times\ \left(\frac{m_1}{a_1}b_{m_1n_1}a_{m_2n_2}a_{m_3n_3} + \frac{m_2}{a_2}a_{m_1n_1}b_{m_2n_2}a_{m_3n_3} + \right.$$

$$\left. + \frac{m_3}{a_3}a_{m_1n_1}a_{m_2n_2}b_{m_3n_3}\right)\ P_{m_1m_2m_3}(t)\ .$$

Let us use (26) to remove term with $\left(\frac{2\pi m_j}{\alpha a_j}\right)^2$

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}(n_1+n_2+n_3)A_{m_1m_2m_3n_1n_2n_3}(t) = \tag{29}$$

$$= 2\pi i\ \left(\frac{1}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}\times$$

$$\times\ \left(\frac{m_1}{a_1}b_{m_1n_1}a_{m_2n_2}a_{m_3n_3} + \frac{m_2}{a_2}a_{m_1n_1}b_{m_2n_2}a_{m_3n_3} + \right.$$

$$\left. + \frac{m_3}{a_3}a_{m_1n_1}a_{m_2n_2}b_{m_3n_3}\right)\ P_{m_1m_2m_3}(t)\ \ = 0.$$

We can easily calculate the sums on n_i in RHS of (29). For this purpose let us introduce the partial sum on the group of terms with constant $(n_1+n_2+n_3) = J$.

$$S_J = 2\pi i\ \left(\frac{\alpha}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\sum_{n_1+n_2+n_3=J} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}\times \tag{30}$$

$$\times a_{m_1n_1}a_{m_2n_2}a_{m_3n_3}\left(\frac{m_1}{a_1}\frac{b_{m_1n_1}}{a_{m_1n_1}} + \frac{m_2}{a_2}\frac{b_{m_2n_2}}{a_{m_2n_2}} + \frac{m_3}{a_3}\frac{b_{m_3n_3}}{a_{m_3n_3}}\right)\ .$$

Let us substitute to (30) values of coefficients a_{ij}, b_{ij}, c_{ij}

$$S_J = \left(\frac{\alpha^2}{k}\right)\ \left(\frac{\alpha}{2\pi k}\right)^{3/2}\sqrt{\frac{2\pi k}{a_1}}\sqrt{\frac{2\pi k}{a_2}}\sqrt{\frac{2\pi k}{a_3}}\left(\frac{k}{\alpha}\right)^J\times \tag{31}$$

$$\times\exp\left[\frac{k}{\alpha}\left(\frac{2\pi m_1}{\alpha a_1}\right)^2\right]\ \exp\left[\frac{k}{\alpha}\left(\frac{2\pi m_2}{\alpha a_2}\right)^2\right]\ \exp\left[\frac{k}{\alpha}\left(\frac{2\pi m_3}{\alpha a_3}\right)^2\right]\ \times$$

$$\times \left(\frac{k}{\alpha} \left[\left(\frac{2\pi m_1}{\alpha a_1} \right)^2 + \left(\frac{2\pi m_2}{\alpha a_2} \right)^2 + \left(\frac{2\pi m_3}{\alpha a_3} \right)^2 \right] + J \right) \times$$

$$\times \sum_{n_1+n_2+n_3=J} \frac{1}{n_1!} \frac{1}{n_2!} \frac{1}{n_3!} \left(\frac{2\pi i m_1}{\alpha a_1} \right)^{2n_1} \left(\frac{2\pi i m_2}{\alpha a_2} \right)^{2n_2} \left(\frac{2\pi i m_3}{\alpha a_3} \right)^{2n_3}.$$

The last sum according to Newton's binomial theorem is

$$S_J = \left(\frac{\alpha^2}{k} \right) \left(\frac{\alpha}{2\pi k} \right)^{3/2} \sqrt{\frac{2\pi k}{a_1}} \sqrt{\frac{2\pi k}{a_2}} \sqrt{\frac{2\pi k}{a_3}} \times \tag{32}$$

$$\times \exp \left[\frac{k}{\alpha} \left(\frac{2\pi m_1}{\alpha a_1} \right)^2 \right] \exp \left[\frac{k}{\alpha} \left(\frac{2\pi m_2}{\alpha a_2} \right)^2 \right] \exp \left[\frac{k}{\alpha} \left(\frac{2\pi m_3}{\alpha a_3} \right)^2 \right] \times$$

$$\times \left(\frac{k}{\alpha} \left[\left(\frac{2\pi m_1}{\alpha a_1} \right)^2 + \left(\frac{2\pi m_2}{\alpha a_2} \right)^2 + \left(\frac{2\pi m_3}{\alpha a_3} \right)^2 \right] + J \right) \times$$

$$\times \frac{1}{J}! \left[\left(\frac{k}{\alpha} \right) \left(\left(\frac{2\pi i m_1}{\alpha a_1} \right)^2 + \left(\frac{2\pi i m_2}{\alpha a_2} \right)^2 + \left(\frac{2\pi i m_3}{\alpha a_3} \right)^2 \right) \right]^J.$$

We see, that result S_J depends on two variables $J = n_1 + n_2 + n_3$ and M

$$M = \left(\frac{k}{\alpha} \right) \left(\left(\frac{2\pi m_1}{\alpha a_1} \right)^2 + \left(\frac{2\pi m_2}{\alpha a_2} \right)^2 + \left(\frac{2\pi m_3}{\alpha a_3} \right)^2 \right). \tag{33}$$

With this variables (32) reads

$$S_J = \left(\frac{\alpha^2}{k} \right) \left(\frac{\alpha}{2\pi k} \right)^{3/2} \sqrt{\frac{2\pi k}{a_1}} \sqrt{\frac{2\pi k}{a_2}} \sqrt{\frac{2\pi k}{a_3}} \times \tag{34}$$

$$\times e^M (M + J) \frac{1}{J}! (-M)^J.$$

The last effort is to calculate sum on J

$$\sum_{J=0}^{\infty} S_J = \left(\frac{\alpha^2}{k} \right) \left(\frac{\alpha}{2\pi k} \right)^{3/2} \sqrt{\frac{2\pi k}{a_1}} \sqrt{\frac{2\pi k}{a_2}} \sqrt{\frac{2\pi k}{a_3}} \times \tag{35}$$

$$\times e^M \sum_{J=0}^{\infty} (M + J) \frac{1}{J}! (-M)^J;$$

$$\sum_{J=0}^{\infty} (M + J) \frac{1}{J}! (-M)^J = M e^{-M} + (-M) e^{-M} = 0. \tag{36}$$

that is coefficient by $P_{m_1 m_2 m_3}(t)$ in (29) is zero and (29) reads

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} c_{m_1 n_1} c_{m_2 n_2} c_{m_3 n_3} (n_1 + n_2 + n_3) A_{m_1 m_2 m_3 n_1 n_2 n_3}(t) = 0. \tag{37}$$

This result is a little disappointment, because we still not get desired equation for $P_{m_1 m_2 m_3}(t)$. We need insistence to achieve success. The result is already near.

Differentiate (37) again on t

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}(n_1+n_2+n_3)A'_{m_1m_2m_3n_1n_2n_3}(t)=0. \tag{38}$$

and substitute A' from (22)

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}(n_1+n_2+n_3)\sum_{j=1}^{j=3}\left[\alpha n_j + k\,\left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\ A_{m_1m_2m_3n_1n_2n_3}(t)= \tag{39}$$

$$=2\pi i\,\left(\frac{\alpha}{k}\right)\,\left(\frac{\alpha}{2\pi k}\right)^{3/2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}(n_1+n_2+n_3)\times$$

$$\times\left(\frac{m_1}{a_1}b_{m_1n_1}a_{m_2n_2}a_{m_3n_3}+\frac{m_2}{a_2}a_{m_1n_1}b_{m_2n_2}a_{m_3n_3}+\right.$$

$$\left.+\frac{m_3}{a_3}a_{m_1n_1}a_{m_2n_2}b_{m_3n_3}\right)\ P_{m_1m_2m_3}(t)\ .$$

Let us use (37) to remove terms with $\left(\frac{2\pi m_j}{\alpha a_j}\right)^2$ in (39). To calculate coefficient before $P_{m_1m_2m_3}(t)$ we perform once again summation on group of terms with constant $(n_1+n_2+n_3)=J$. The sum for each group is calculated as before, but this time this sum is multiplied by J before final summation on J

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}\alpha(n_1+n_2+n_3)^2A_{m_1m_2m_3n_1n_2n_3}(t)= \tag{40}$$

$$=\left(\frac{\alpha^2}{k}\right)\,\left(\frac{\alpha}{2\pi k}\right)^{3/2}\sqrt{\frac{2\pi k}{a_1}}\sqrt{\frac{2\pi k}{a_2}}\sqrt{\frac{2\pi k}{a_3}}e^M\sum_{J=0}^{\infty}(M+J)\,\frac{J}{J}!(-M)^JP_{m_1m_2m_3}(t)\ .$$

This time the sum on J is not zero

$$\sum_{J=0}^{\infty}(M+J)\,\frac{J}{J}!(-M)^J=-M^2e^{-M}+(-M)^2e^{-M}-Me^{-M}=-Me^{-M}, \tag{41}$$

and we get finally equation for pressure, which follows from incompressibility condition

$$\boxed{\begin{gathered}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1n_1}c_{m_2n_2}c_{m_3n_3}(n_1+n_2+n_3)^2A_{m_1m_2m_3n_1n_2n_3}(t)= \qquad (42)\\ =-\left(\frac{\alpha}{k}\right)\,\left(\frac{\alpha}{2\pi k}\right)^{3/2}\sqrt{\frac{2\pi k}{a_1}}\sqrt{\frac{2\pi k}{a_2}}\sqrt{\frac{2\pi k}{a_3}}MP_{m_1m_2m_3}(t)\ .\end{gathered}}$$

(42) is algebraic equation, but in the same time it is Fourier transform of some differential equation for original unknown variable n. Namely according to (33) $M=\left(\frac{k}{\alpha}\right)\,\left(\left(\frac{2\pi m_1}{\alpha a_1}\right)^2+\left(\frac{2\pi m_2}{\alpha a_2}\right)^2+\left(\frac{2\pi m_3}{\alpha a_3}\right)^2\right)$. Each m_i^2 term in (42) is Fourier transform of second partial derivative of p on corresponding space

variable, their sum is Fourier transform of Laplace operator. Therefore (42) is Fourier transform of Poisson's equation for pressure.

Solve this equation for $P_{m_1 m_2 m_3}(t)$ and get

$$P_{m_1 m_2 m_3}(t) = \frac{-1}{M}\left(\frac{k}{\alpha}\right)\ \left(\frac{2\pi k}{\alpha}\right)^{3/2}\sqrt{\frac{a_1}{2\pi k}}\sqrt{\frac{a_2}{2\pi k}}\sqrt{\frac{a_3}{2\pi k}}\times \tag{43}$$

$$\times\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} c_{m_1 n_1} c_{m_2 n_2} c_{m_3 n_3} (n_1+n_2+n_3)^2 A_{m_1 m_2 m_3 n_1 n_2 n_3}(t).$$

Let us take values of c_{mn} from (25)

$$P_{m_1 m_2 m_3}(t) = \frac{-\exp(-M/2)}{M}\left(\frac{k}{\alpha}\right)\ \left(\frac{2\pi k}{\alpha}\right)^{3/2}\times \tag{44}$$

$$\times\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\left(\frac{2k}{\alpha}\right)^{(\nu_1+\nu_2+\nu_3)/2}\left(\frac{2\pi i m_1}{\alpha a_1}\right)^{\nu_1}\left(\frac{2\pi i m_2}{\alpha a_2}\right)^{\nu_2}\left(\frac{2\pi i m_3}{\alpha a_3}\right)^{\nu_3}(\nu_1+\nu_2+\nu_3)^2 A_{m_1 m_2 m_3 \nu_1 \nu_2 \nu_3}(t).$$

and substitute result to (23)

$$\frac{d}{dt}A_{m_1 m_2 m_3 n_1 n_2 n_3}(t) + \sum_{j=1}^{j=3}\left[\alpha n_j + k\ \left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\ A_{m_1 m_2 m_3 n_1 n_2 n_3}(t) = \tag{45}$$

$$= 2\pi i\ a_{m_1 n_1} a_{m_2 n_2} a_{m_3 n_3}\ \left(\frac{m_1}{a_1}\frac{b_{m_1 n_1}}{a_{m_1 n_1}} + \frac{m_2}{a_2}\frac{b_{m_2 n_2}}{a_{m_2 n_2}} + \frac{m_3}{a_3}\frac{b_{m_3 n_3}}{a_{m_3 n_3}}\right)\ \frac{\exp(-M/2)}{(-M)}\times$$

$$\times\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\left(\frac{2k}{\alpha}\right)^{(\nu_1+\nu_2+\nu_3)/2}\left(\frac{2\pi i m_1}{\alpha a_1}\right)^{\nu_1}\left(\frac{2\pi i m_2}{\alpha a_2}\right)^{\nu_2}\left(\frac{2\pi i m_3}{\alpha a_3}\right)^{\nu_3}(\nu_1+\nu_2+\nu_3)^2 A_{m_1 m_2 m_3 \nu_1 \nu_2 \nu_3}(t).$$

To get final result, substitute values of a_{mn} from (17)

$$\frac{d}{dt}A_{m_1 m_2 m_3 n_1 n_2 n_3}(t) + \sum_{j=1}^{j=3}\left[\alpha n_j + k\ \left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\ A_{m_1 m_2 m_3 n_1 n_2 n_3}(t) = \tag{46}$$

$$= \frac{\exp(M)}{(-M)}\frac{1}{2^{n_1+n_2+n_3} n_1! n_2! n_3!}\left(\frac{-2k}{\alpha}\right)^{(n_1+n_2+n_3)/2}\times$$

$$\times\ \left(\frac{2\pi m_1}{\alpha a}\right)^{n_1}\left(\frac{2\pi m_2}{\alpha a}\right)^{n_2}\left(\frac{2\pi m_3}{\alpha a}\right)^{n_3}\sum_{j=1}^{j=3}\left[\alpha n_j + k\ \left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]\times$$

$$\times\sum_{\nu_1=0}^{\infty}\sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\left(\frac{-2k}{\alpha}\right)^{(\nu_1+\nu_2+\nu_3)/2}\left(\frac{2\pi m_1}{\alpha a_1}\right)^{\nu_1}\left(\frac{2\pi m_2}{\alpha a_2}\right)^{\nu_2}\left(\frac{2\pi m_3}{\alpha a_3}\right)^{\nu_3}(\nu_1+\nu_2+\nu_3)^2 A_{m_1 m_2 m_3 \nu_1 \nu_2 \nu_3}(t).$$

We totally eliminated pressure from dynamic equation. Thus the problem is reduced to the system of ordinary linear differential equations (46). Such systems are solved by Euler's exponential substitution. The only difficulty is, that the number of variables is indefinitely great. We try to integrate (46) using special properties of its matrix.

6 Special eigenvalue problem

In this section we consider purely algebraic eigenvalue problem for special form of matrices.

Let the matrix have the following form

$$M_{ij} = D_{ij} + P_{ij}; \tag{47}$$

where D_{ij} - diagonal matrix

$$D_{ij} = \begin{cases} 0; & i \neq j \\ d_{i;} & i = j \end{cases} \tag{48}$$

P_{ij} - diadic product matrix

$$P_{ij} = l_i r_j. \tag{49}$$

The eigenvalue problem consists of following tasks:

1) To find a set of eigenvectors x such, that matrix product of M and x is proportional to x

$$M_{ij} x_i = \lambda x_j. \tag{50}$$

2) To find corresponding set of proportionality coefficients - eigenvalues λ.

For our special form of matrix

$$(D_{ij} - \lambda\ \delta_{ij}) x_j + l_i\ (r_k x_k) = 0; \tag{51}$$

Let us denote

$$S = r_k x_k; \tag{52}$$

Then

$$(d_1 - \lambda)\ \frac{x_1}{l_1} = (d_2 - \lambda)\ \frac{x_2}{l_2} = ... = -S. \tag{53}$$

This gives very simple expression for components of eigenvector, provided eigenvalue λ is evident

$$x_i = \frac{-S l_i}{(d_i - \lambda)}. \tag{54}$$

Substitute this expression to (52) and get

$$\sum_k \frac{-S\ l_k r_k}{(d_k - \lambda)} = S. \tag{55}$$

Two cases are possible here. The common case is $S! = 0$.

$$\sum_k \frac{l_k r_k}{(d_k - \lambda)} + 1 = 0. \tag{56}$$

(56) gives equation for λ. When all d_i are different, algebraic equation (56) has degree n, where n - dimension of matrix M.

If all roots of (56) are different, we get the full set of eigenvalues, then from (54) we find full set of eigenvectors. Exact value of S is of no meaning, we can put for example $S = 1$ in (54).

For example, when all $(l_k r_k)$ are positive or all are negative and all d_i are real (and different - see above), one could guarantee that all n roots of (56) are real and different. This follows from the fact, that d_i separate the roots of (56). Therefore there are $n-1$ roots between d_i and one more root $x_1 min(d_i)$ (the case $l_i r_i 0$). This fact makes numeric evaluation of roots rather simple. Unfortunately we deal with quite opposite case - signs of our $(l_k r_k)$ are alternating. Nevertheless we shall find roots - see below.

When some d_i are equal, equation (56) has less roots then matrix degree. We can consider this case as confluent. In this case additional eigenvalues λ (besides roots of (56)) must be equal to iterated value d_i and we must put for these eigenvalues $S = 0$ in (53). Then all x_i , besides that in columns, corresponding to iterated d_i, must be zero. The rest nonzero x_i must satisfy orthogonality condition (52) (with $S = 0$).

Let us consider the eigenvalue problem for transposed matrix M^T . Let us denote y - eigenvectors for M^T. Then

$$(D_{ij} - \lambda\ \delta_{ij}) y_i + (l_k y_k) r_j = 0; \tag{57}$$

or

$$(a_i - \lambda)\frac{y_i}{r_i} = -S. \tag{58}$$

$$y_i = \frac{-S r_i}{(d_i - \lambda)}. \tag{59}$$

$$S = l_k y_k = \sum_k \frac{-S\ l_k r_k}{(d_k - \lambda)}. \tag{60}$$

We get for λ equation (56) once again - eigenvalues of conjugated problems are equal. Components of eigenvectors for conjugated problem are calculated from (59), where S is arbitrary nonzero number, for example $S = 1$.

Eigenvector for conjugated problems with different eigenvalues λ and μ are orthogonal:

$$\sum_k x_k y_k = \sum_k \frac{-l_k}{(d_k - \lambda)} \frac{-r_k}{(d_k - \mu)} = \sum_k \frac{l_k r_k}{\lambda - \mu} \left(\frac{1}{(d_k - \lambda)} - \frac{1}{(d_k - \mu)} \right) = \tag{61}$$

$$= \frac{1}{\lambda - \mu} \left(\sum_k \frac{l_k r_k}{(d_k - \lambda)} - \sum_k \frac{l_k r_k}{(d_k - \mu)} \right) = \frac{1}{\lambda - \mu} (-1 + 1) = 0.$$

7 Application to equation (46)

Let us return to our equation (46). Comparing with previous section, we could identify matrix components in the following way.

Diagonal components of the matrix are equal to:

$$d_{n_1 n_2 n_3} = \sum_{j=1}^{j=3} \left[\alpha n_j + k\ \left(\frac{2\pi m_j}{\alpha a_j} \right)^2 \right]. \tag{62}$$

Diagonal components depend only on M (see (33)) and $J = \sum n_l$. Therefore we deal with confluent case - see previous section. Some eigenvalues are equal to $d_{n_1 n_2 n_3}$.

Nondiagonal components of the matrix are equal to:

$$l_{n_1n_2n_3} = \frac{\exp(M)}{M}\frac{1}{2^{n_1+n_2+n_3}n_1!n_2!n_3!}\left(\frac{-2k}{\alpha}\right)^{(n_1+n_2+n_3)/2} \times \tag{63}$$

$$\times \left(\frac{2\pi m_1}{\alpha a}\right)^{n_1}\left(\frac{2\pi m_2}{\alpha a}\right)^{n_2}\left(\frac{2\pi m_3}{\alpha a}\right)^{n_3}\sum_{j=1}^{j=3}\left[\alpha n_j + k\left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right].$$

$$r_{\nu_1\nu_2\nu_3} = \left(\frac{-2k}{\alpha}\right)^{(\nu_1+\nu_2+\nu_3)/2}\left(\frac{2\pi m_1}{\alpha a_1}\right)^{\nu_1}\left(\frac{2\pi m_2}{\alpha a_2}\right)^{\nu_2}\left(\frac{2\pi m_3}{\alpha a_3}\right)^{\nu_3}(\nu_1+\nu_2+\nu_3)^2. \tag{64}$$

In all cases we replaced index (i) with multiindex $(n_1n_2n_3)$.

Characteristic equation for λ is

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{l_{n_1n_2n_3}r_{n_1n_2n_3}}{\lambda - d_{n_1n_2n_3}} = \tag{65}$$

$$= \frac{e^M}{M}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\frac{\sum_{j=1}^{j=3}\left[\alpha n_j + k\left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]}{\lambda - \sum_{j=1}^{j=3}\left[\alpha n_j + k\left(\frac{2\pi m_j}{\alpha a_j}\right)^2\right]}\times$$

$$\times \left(\frac{-k}{\alpha}\right)^{(n_1+n_2+n_3)}\left(\frac{2\pi m_1}{\alpha a_1}\right)^{2n_1}\left(\frac{2\pi m_2}{\alpha a_2}\right)^{2n_2}\left(\frac{2\pi m_3}{\alpha a_3}\right)^{2n_3}\frac{(n_1+n_2+n_3)^2}{n_1!n_2!n_3!} = 1.$$

Let us once again calculate partial sum S_J of terms, for which $n_1+n_2+n_3 = J$. We have

$$S_J = \frac{e^M}{M}\left(\frac{\alpha(J+M)J^2}{\lambda - \alpha(J+M)}\right)\frac{(-M)^J}{J}!. \tag{66}$$

$$\sum_{J=0}^{\infty}S_J = (1-M) - \frac{\lambda}{\alpha}\frac{e^M}{M}M^{-(M-\lambda/\alpha)}\times \tag{67}$$

$$\times\ [\gamma((2+M-\lambda/\alpha), M) - \gamma((1+M-\lambda/\alpha), M)] = 1.$$

where $\gamma(p,z)$ is incomplete gamma - function (see [4], [5])

$$\gamma(p,z) = \int_{t=0}^{t=z} e^{-t}t^{p-1}dt. \tag{68}$$

(65) reads now

$$M + \frac{\lambda}{\alpha}\frac{e^M}{M}M^{-(M-\lambda/\alpha)}\ [\gamma((2+M-\lambda/\alpha), M) - \gamma((1+M-\lambda/\alpha), M)] = 0. \tag{69}$$

We see, that all eigenvalues λ are proportional to α

$$\lambda = \xi\alpha, \tag{70}$$

where coefficients ξ are roots of equation

$$M+\xi\frac{e^M}{M}M^{-(M-\xi)}\ \left[\gamma((2+M-\xi),M)-\gamma((1+M-\xi),M)\right]=0. \tag{71}$$

We can further simplify (71) using known recurrence (see [4], 9.2)

$$\gamma(p+1,z)=p\gamma(p,z)-x^pe^{-x}. \tag{72}$$

$$\begin{aligned} M+\xi\frac{e^M}{M}M^{-(M-\xi)}\ &\left[(1+M-\xi)\gamma((1+M-\xi),M)-\right. \\ &\left.-M^{(1+M-\xi)}e^{-M}-\gamma((1+M-\xi),M)\right]=0. \end{aligned} \tag{73}$$

$$M+\xi\frac{e^M}{M}M^{-(M-\xi)}\ \left[(M-\xi)\gamma((1+M-\xi),M)-M^{(1+M-\xi)}e^{-M}\right]=0. \tag{74}$$

$$M-\xi+\xi\frac{e^M}{M}M^{-(M-\xi)}\ (M-\xi)\gamma((1+M-\xi),M)=0. \tag{75}$$

One exact root of equation (75) we can find easily - this is root

$$\xi=M. \tag{76}$$

Another roots satisfy reduced equation

$$1+\xi\frac{e^M}{M}M^{-(M-\xi)}\ \gamma((1+M-\xi),M)=0. \tag{77}$$

Another form of (77) we obtain using modified incomplete gamma function

$$\gamma^*(p,x)=\frac{x^{-p}}{\Gamma(p)}\gamma(p,x). \tag{78}$$

This form is

$$1+\xi e^M\Gamma(1+M-\xi)\ \gamma^*((1+M-\xi),M)=0. \tag{79}$$

Advantage of this form is that $\gamma^*(p,x)$ is a single valued analytic function of p and x possessing no finite singularities.

We calculate some roots of equation (77) using definition of incomplete gamma-function (66 - 67). Namely we keep only finite number of terms in the sum (66 - 67) and solve resulting algebraic equation using Newton's method. As the number of terms increase, the roots converge rapidly.

As initial approximation for roots we use position of poles, that is we search roots in the close vicinity of the pole. We keep corresponding term and approximate contribution of the rest poles by two first terms of Taylor series. So we have quadratic equation. When this equation has two conjugated roots, Newton's iterations converge to two conjugated roots of full equation. When quadratic equation has two real roots, one root is really located in the poles vicinity and another is far enough, so that Newton's iterations diverge.

The results of our calculations are presented in the Table 1 (see APPENDIX). We see, that:

- All roots have positive real part. Therefore exists root with the least real part and roots can be ordered according their real part in ascending order. In the following we suppose that such ordering is done.

- For each M only finite number of roots are complex and imaginary part of roots decreases with root number.

- Starting from some root all roots are real (tail).

- Real roots are located very close to poles (natural numbers) and rapidly get indistinguishable.

On this stage we shall content ourself by these experimental results and shall not try to give them rigorous proof. The theory of distribution of roots of analytic functions in question is rather ample (see [6], [7], [8]).

As we see from previous section, besides roots of (77) there exist eigenvalues, which are exactly equal to diagonal values $d_{n_1n_2n_3}$ (62). They correspond to zero-pressure solutions from our work [2]. Orthogonality conditions (52), which must be satisfied by eigenfunctions, according to (64) and (43) mean, that $P_{m_1m_2m_3}(t) = 0$.

8 Solution of Cauchy problem

In this section we describe construction of solution of linearized Fokker - Planck equation for incompressible fluid.

1) To setup Cauchy problem we must set initial value of n variable. This initial value $n_0 = n_0(x_1, x_2, x_3, v_1, v_2, v_3)$ must satisfy incompressibility condition (1). There is no need to set initial value of pressure, because it is fully determined by n (see (42)).

2) Calculate Fourier coefficients $A_{m_1m_2m_3p_1p_2p_3}(0)$ (see [3] for details):

$$A_{m_1m_2m_3p_1p_2p_3}(0) = \frac{1}{2^{p_1+p_2+p_3}p_1!p_2!p_3!} \left(\frac{\alpha}{2\pi k}\right)^{\frac{3}{2}} \frac{1}{a_1a_2a_3}\times$$
$$\times \exp\left[\frac{\alpha}{2k}\left(\frac{4\pi k}{\alpha^2}\right)^2 \left(\left(\frac{m_1}{a_1}\right)^2 + \left(\frac{m_2}{a_2}\right)^2 + \left(\frac{m_3}{a_3}\right)^2\right)\right]\times$$
$$\times \int_0^{a_1} dx_1 \int_0^{a_2} dx_2 \int_0^{a_3} dx_3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n_0(x_1, x_2, x_3, v_1, v_2, v_3) \ \ \psi_{m_1m_2m_3p_1p_2p_3} dv_1dv_2dv_3. \tag{80}$$

where

$$\psi_{m_1m_2m_3n_1n_2n_3} = \prod_{j=1}^{j=3} \exp\left(-2\pi i\frac{m_j}{a}(x_j + \frac{v_j}{\alpha})\right) \ H_{n_j}\left(\sqrt{\frac{\alpha}{2k}}\left(v_j + \frac{4\pi i m_j k}{\alpha^2 a_j}\right)\right). \tag{81}$$

3) Change from eigenfunctions of simple Fokker - Planck equations to eigenfunctions of linearized Fokker - Planck equation for incompressible fluid. According to (61) projection of vector A_k on eigenvector x_μ, corresponding to eigenvalue λ, is

$$A_\lambda = \frac{\sum_k A_k y_k}{\sum_k y_k y_k} = \left(\sum_k A_k \frac{r_k}{(\lambda - d_k)}\right) \ / \ \left(\sum_k \frac{r_k^2}{(\lambda - d_k)^2}\right) \tag{82}$$

or according to (64)

$$A_{m_1m_2m_3\lambda}(0) = \left(\sum_{\nu_1\nu_2\nu_3} A_{m_1m_2m_3\nu_1\nu_2\nu_3}(0) \, \left(\frac{-2k}{\alpha}\right)^{(\nu_1+\nu_2+\nu_3)/2} \times \right. \tag{83}$$

$$\times \left(\frac{2\pi m_1}{\alpha a_1}\right)^{\nu_1} \left(\frac{2\pi m_2}{\alpha a_2}\right)^{\nu_2} \left(\frac{2\pi m_3}{\alpha a_3}\right)^{\nu_3} \times$$

$$\left. \times \frac{(\nu_1+\nu_2+\nu_3)^2}{(\lambda - d_{\nu_1\nu_2\nu_3})} \right) \quad / \quad \left(\sum_k \frac{r^2_{n_1n_2n_3}}{(\lambda - d_{\nu_1\nu_2\nu_3})^2} \right)$$

where $d_{n_1n_2n_3}$ is defined by (62), $r_{n_1n_2n_3}$ is defined by (64).

4) Initial field can contain some zero pressure solutions. Let us suppose, that there exist a group of coefficients with constant $J = \sum \nu_l$, for which

$$\sum_{\nu_1+\nu_2+\nu_3=J} A_{m_1m_2m_3\nu_1\nu_2\nu_3}(0) \, \left(\frac{2\pi m_1}{\alpha a_1}\right)^{\nu_1} \left(\frac{2\pi m_2}{\alpha a_2}\right)^{\nu_2} \left(\frac{2\pi m_3}{\alpha a_3}\right)^{\nu_3} = 0. \tag{84}$$

This group does not contribute to any $A_{m_1m_2m_3\lambda}$ (see 83). Such groups, if present, we must consider separately. They correspond to zero pressure solutions of our work [2].

5) Given initial values of A_λ we can calculate their values for the arbitrary moment t according to exponential law

$$A_{m_1m_2m_3\lambda}(t) = e^{-\lambda t} A_{m_1m_2m_3\lambda}(0). \tag{85}$$

6) Evolution of zero pressure solution is determined by exponential multiplier $e^{-\alpha(J+M)t}$.

7) Inverse transition from A_λ to A_k is

$$A_k = \sum_\lambda A_\lambda \frac{l_k}{(\lambda - d_k)} \tag{86}$$

or according to (63)

$$A_{m_1m_2m_3n_1n_2n_3}(t) = \sum_\lambda A_{m_1m_2m_3\lambda}(t) \frac{\exp(M)}{M} \times \tag{87}$$

$$\times \frac{1}{2^{n_1+n_2+n_3} n_1! n_2! n_3!} \left(\frac{-2k}{\alpha}\right)^{(n_1+n_2+n_3)/2} \times$$

$$\times \, \left(\frac{2\pi m_1}{\alpha a}\right)^{n_1} \left(\frac{2\pi m_2}{\alpha a}\right)^{n_2} \left(\frac{2\pi m_3}{\alpha a}\right)^{n_3} \sum_{j=1}^{j=3} \left[\alpha n_j + k \, \left(\frac{2\pi m_j}{\alpha a_j}\right)^2 \right] \, \frac{1}{(\lambda - d_{n_1n_2n_3})}.$$

$d_{n_1n_2n_3}$ defined by (62).

8) Contribution from zero pressure solutions must be added to (87).

9) Pressure for each moment of time is determined by (44).

DISCUSSION

We see that spectral properties of Fokker - Planck linearized differential operator for incompressible fluid are different from properties of usual operator. General spectrum structure is roughly similar, but nearest to zero eigenvalues are complex. Therefore most slowly damping modes are most strongly vibrating - very interesting result. Generally all modes are damping with time, flows tend to the rest.

REFERENCES

[1] Igor A. Tanski. Fokker - Planck equation for incompressible fluid. arXiv:0812.2303v2 [nlin.CD] 3 Feb 2009

[2] Igor A. Tanski. Two simple solutions of nonlinear Fokker - Planck equation for incompressible fluid. arXiv:0812.4795v2 [nlin.CD] 25 Feb 2009

[3] Igor A. Tanski. Spectral decomposition of 3D Fokker - Planck differential operator. arXiv:nlin/0607050v3 [nlin.CD] 25 Jun 2007

[4] H. Bateman, A. Erdelyi. Higher transcendental functions. vol. 2, Mc Graw-Hill, New York, 1953

[5] M. Abramovitz, I. A. Stegun, Handbook of Mathematical Functions. National Bureau of Standards, 1970

[6] K. S. Koelbig. On the zeros of incomplete gamma function. Mathematics of Computation, vol. 26, num. 119, Jul 1972

[7] Walter Gautschi. The incomplete gamma functions since Tricomi. In Tricomi's Ideas and Contemporary Applied Mathematics, Atti dei Convegni Lincei, n. 147, Accademia Nazionale dei Lincei

[8] A. M. Sedlecki. Zeros of Mittag-Leffler functions. Matematicheskie zametki, vol. 68, num. 5, Nov. 2000

APPENDIX 1

Roots of equation (71)

Root $\xi = M$ is omitted

NN	M=1	M=2	M=3
1	3.84958810 - 1.92315575 i	4.52745332 - 3.27206660 i	5.04119504 - 4.34568739 i
2	3.84958810 + 1.92315575 i	4.52745332 + 3.27206660 i	5.04119504 + 4.34568739 i
3	5.94063198 - 1.14455587 i	7.06814092 - 2.67480339 i	7.90915778 - 3.92341797 i
4	5.94063198 + 1.14455587 i	7.06814092 + 2.67480339 i	7.90915778 + 3.92341797 i
5	7.69165960 - 0.29172595 i	9.14966499 - 1.97226724 i	10.24525406 - 3.33703003 i
6	7.69165960 + 0.29172595 i	9.14966499 + 1.97226724 i	10.24525406 + 3.33703003 i
7	11.000932211	11.00141191 - 1.21959567 i	12.31747194 - 2.67114557 i
8	11.999889388	11.00141191 + 1.21959567 i	12.31747194 + 2.67114557 i
9	13.000011755	12.72299911 - 0.42498830 i	14.22286668 - 1.95786294 i
10	13.999998866	12.72299911 + 0.42498830 i	14.22286668 + 1.95786294 i
11	15.000000099	14.07454792	16.00982241 - 1.21325010 i
12	15.999999999	14.98376860	16.00982241 + 1.21325010 i
13	17.000000000	16.00272440	17.72151356 - 0.43464100 i
14	17.999999999	16.99956146	17.72151356 + 0.43464100 i
15	19.000000000	18.00006506	19.98044555
16	19.999999999	18.99999098	19.08192627
17	21.000000000	20.00000116	19.98044554
18	22.000000000	20.99999985	21.00363177
19	23.000000000	22.00000001	21.99933574
20	24.000000000	22.99999999	23.00011373

NN	M=4	M=5	M=6
1	5.47498406 - 5.26768710 i	5.85889775 - 6.08913226 i	6.20771808 - 6.83731700 i
2	5.47498406 + 5.26768710 i	5.85889775 + 6.08913226 i	6.20771808 + 6.83731700 i
3	8.61344667 - 5.01144505 i	9.23406886 - 5.99054148 i	9.79675137 - 6.88888105 i
4	8.61344667 + 5.01144505 i	9.23406886 + 5.99054148 i	9.79675137 + 6.88888105 i
5	11.15620212 - 4.53812831 i	11.95509246 - 5.62665552 i	12.67703265 - 6.63086922 i
6	11.15620212 + 4.53812831 i	11.95509246 + 5.62665552 i	12.67703265 + 6.63086922 i
7	13.40577133 - 3.95833350 i	14.35634208 - 5.13111612 i	15.21274497 - 6.21756080 i
8	13.40577133 + 3.95833350 i	14.35634208 + 5.13111612 i	15.21274497 + 6.21756080 i
9	15.47089306 - 3.31449740 i	16.55737228 - 4.55585073 i	17.53366817 - 5.70963951 i
10	15.47089306 + 3.31449740 i	16.55737228 + 4.55585073 i	17.53366817 + 5.70963951 i
11	17.40539050 - 2.62778318 i	18.61704841 - 3.92690815 i	19.70341924 - 5.13774003 i
12	17.40539050 + 2.62778318 i	18.61704841 + 3.92690815 i	19.70341924 + 5.13774003 i
13	19.24064677 - 1.91036878 i	20.56955610 - 3.25932488 i	21.75880445 - 4.51960861 i
14	19.24064677 + 1.91036878 i	20.56955610 + 3.25932488 i	21.75880445 + 4.51960861 i
15	20.99699656 - 1.17010968 i	22.43688595 - 2.56260936 i	23.72343062 - 3.86649261 i
16	20.99699656 + 1.17010968 i	22.43688595 + 2.56260936 i	23.72343062 + 3.86649261 i
17	22.70576400 - 0.39532166 i	24.23420403 - 1.84317596 i	25.61354419 - 3.18599046 i
18	22.70576400 + 0.39532166 i	24.23420403 + 1.84317596 i	25.61354419 + 3.18599046 i
19	24.07528691	25.97268205 - 1.10579718 i	27.44091387 - 2.48348667 i
20	24.98146932	25.97268205 + 1.10579718 i	27.44091387 + 2.48348667 i

#	M=7	M=8	M=9
1	6.53001156 - 7.52895368 i	6.83127967 - 8.17518985 i	7.11531330 - 8.78391538 i
2	6.53001156 + 7.52895368 i	6.83127967 + 8.17518985 i	7.11531330 + 8.78391538 i
3	10.31618762 - 7.72399088 i	10.80167947 - 8.50772212 i	11.25955129 - 9.24856515 i
4	10.31618762 + 7.72399088 i	10.80167947 + 8.50772212 i	11.25955129 + 9.24856515 i
5	13.34199931 - 7.56851364 i	13.96257983 - 8.45167276 i	14.54728640 - 9.28905550 i
6	13.34199931 + 7.56851364 i	13.96257983 + 8.45167276 i	14.54728640 + 9.28905550 i
7	15.99974275 - 7.23543754 i	16.73291571 - 8.19691791 i	17.42277980 - 9.11081249 i
8	15.99974275 + 7.23543754 i	16.73291571 + 8.19691791 i	17.42277980 + 9.11081249 i
9	18.42897216 - 6.79355442 i	19.26164201 - 7.81977962 i	20.04405940 - 8.79717049 i
10	18.42897216 + 6.79355442 i	19.26164201 + 7.81977962 i	20.04405940 + 8.79717049 i
11	20.69785002 - 6.27781735 i	21.62130874 - 7.35928644 i	22.48793355 - 8.39101050 i
12	20.69785002 + 6.27781735 i	21.62130874 + 7.35928644 i	22.48793355 + 8.39101050 i
13	22.84567896 - 5.70854973 i	23.85362358 - 6.83821861 i	24.79844419 - 7.91745869 i
14	22.84567896 + 5.70854973 i	23.85362358 + 6.83821861 i	24.79844419 + 7.91745869 i
15	24.89759650 - 5.09866262 i	25.98519281 - 6.27108315 i	27.00361964 - 7.39255586 i
16	24.89759650 + 5.09866262 i	25.98519281 + 6.27108315 i	27.00361964 + 7.39255586 i
17	26.87086392 - 4.45689278 i	28.03424697 - 5.66771323 i	29.12262537 - 6.82719343 i
18	26.87086392 + 4.45689278 i	28.03424697 + 5.66771323 i	29.12262537 + 6.82719343 i
19	28.77796298 - 3.78943945 i	30.01394562 - 5.03509475 i	31.16927222 - 6.22911873 i
20	28.77796298 + 3.78943945 i	30.01394562 + 5.03509475 i	31.16927222 + 6.22911873 i